WNE Linear Algebra
Resit Exam
Series A

24 February 2023

Please use separate sheets for different problems. Please use single sheet for all questions. <u>Please include all relevant calculations</u>. Give reasons to your answers. Please provide the following data on each sheet

- **name, surname and your student number,**
- **number of your group,**
- **number of the corresponding problem and the series.**

Each problem is worth 10 marks. Each question is worth 4 marks.

Problems

Problem 1.
Let $V = \mathrm{lin}((1,1,4,1),(1,2,3,3),(1,5,0,9))$ be a subspace of $\mathbb{R}^4$.

a) find a basis and the dimension of the subspace V,
b) find a system of linear equations which set of solutions is equal to V.

Problem 2.
Let $V \subset \mathbb{R}^4$ be a subspace given by the homogeneous system of linear equations

$$\begin{cases} x_1 + 2x_2 + 9x_3 + 17x_4 = 0 \\ x_1 + x_2 + 7x_3 + 11x_4 = 0 \end{cases}$$

a) find a basis $\mathcal{A}$ and the dimension of the subspace V,
b) let $v = (-10, 0, 3t, -t) \in \mathbb{R}^4$. Find all $t \in \mathbb{R}$ such that $v \in V$ (i.e., vector v belongs to V) and compute coordinates of v relative to $\mathcal{A}$.

Problem 3.
Let $\mathcal{A} = ((0,1),(-1,1))$ be a basis of $\mathbb{R}^2$ and let $\mathcal{B} = ((0,0,1),(1,0,-1),(1,1,0))$ be a basis of $\mathbb{R}^3$. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by the matrix

$$M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 2 & 3 \\ 3 & 1 \\ 1 & 2 \end{bmatrix}.$$

Let $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by the formula

$$\psi((x_1, x_2)) = (x_1 + x_2, x_1).$$

a) find the formula of φ.
b) find the matrix $M(\varphi \circ \psi)_{\mathcal{A}}^{\mathcal{B}}$.

Problem 4.
Let $V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2x_1 - x_2 + 2x_3 = 0\}$ be a subspace of $\mathbb{R}^3$.

a) find an orthonormal basis of $V^{\perp}$,
b) compute the orthogonal reflexion/symmetry of $w = (3, 1, 2)$ across V.

Problem 5.a) let $q\colon \mathbb{R}^2 \to \mathbb{R}$ be a quadratic form given by the formula

$$q((x_1, x_2)) = -2x_1^2 + 4x_1x_2 - 3x_2^2.$$

Check if the form q is negative definite.

b) let $Q\colon \mathbb{R}^3 \to \mathbb{R}$ be a quadratic form given by the formula
$$Q((x_1, x_2, x_3)) = x_1^2 + 10x_2^2 + 9x_3^2 + 6x_1x_3.$$
Check if the form Q is positive semidefinite.

Problem 6.
Consider the following linear programming problem $x_1 - x_4 \to \min$ in the standard form with constraints
$$\begin{cases} \quad x_1 - x_2 + x_3 = 1 \\ -2x_1 + 2x_2 + x_4 = 4 \\ -x_1 + x_2 + x_5 = 3 \end{cases} \quad \text{and } x_i \geqslant 0 \text{ for } i = 1, \ldots, 5.$$

a) which of the sets $\mathcal{B}_1 = \{2, 4, 5\},\ \mathcal{B}_2 = \{3, 4, 5\},\ \mathcal{B}_3 = \{1, 2, 5\}$ is basic feasible? write the corresponding basic solution for all basic sets,

b) solve the linear programming problem using simplex method.

Questions

Question 1.
Let $A, B \in M(2 \times 2; \mathbb{R})$ be matrices such that $A^2 = A,\ B^2 = B$ and $AB = BA$. Does it follow that
$$(A + B - AB)^2 = A + B - AB?$$

Solution 1.
Yes, it does.
$$(A+B-AB)^2 = (A+B-AB)(A+B-AB) = A(A+B-AB)+B(A+B-AB)-AB(A+B-AB) =$$
$$= A^2 + AB - A^2B + BA + B^2 - BAB - ABA - AB^2 + ABAB =$$
$$= A + AB - AB + AB + B - AB - AB - AB + AB = A + B - AB.$$

Question 2.
Let $A = [a_{ij}] \in M(3 \times 3; \mathbb{R})$ be a matrix. Let
$$B = \begin{bmatrix} -a_{11} & a_{12} & -a_{13} \\ a_{21} & -a_{22} & a_{23} \\ -a_{31} & a_{32} & -a_{33} \end{bmatrix}.$$
Does it follow that $\det A = \det B$?

Solution 2.
No, it does not. By the Sarrus rule $\det B = -\det A$.

Question 3.
Give an example of a matrix $A \in M(2 \times 2; \mathbb{R})$ with eigenvalues $\lambda = 0$ and $\lambda = -1$ such that
$$V_{(0)} = \operatorname{lin}((1, 0)), \quad V_{(-1)} = \operatorname{lin}((1, -1)).$$
Is matrix A uniquely determined?

Solution 3.
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The conditions are equivalent to
$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ -1 \end{bmatrix},$$
that is
$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} a - b \\ c - d \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

This gives $a = c = 0$, and consequently $b = 1,\ d = -1$. Using non–zero vectors proportional to $(1, 0)$ and $(-1, 1)$, that is $(p, 0)$ and $(-q, q)$ for $p, q \neq 0$ yields the same numbers, therefore matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ is uniquely determined.

Question 4.
Let $A \in M(2 \times 2; \mathbb{R})$ be a symmetric matrix, i.e. $A^\intercal = A$. Does it follow that for any $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$ and any $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}^2$

$$(Av) \cdot w = (Aw) \cdot v,$$

where $v \cdot w$ denotes the standard scalar product of vectors $v, w \in \mathbb{R}^2$?

Solution 4.
Yes, it does. If $A = A^\intercal = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, then

$$(Av) \cdot w = \begin{bmatrix} av_1 + bv_2 & bv_1 + cv_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = av_1w_1 + bv_2w_1 + bv_1w_2 + cv_2w_2,$$

and

$$(Aw) \cdot w = \begin{bmatrix} aw_1 + bw_2 & bw_1 + cw_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = av_1w_1 + bv_2w_1 + bv_1w_2 + cv_2w_2.$$

In general, for any matrix $A = A^\intercal \in M(n \times n; \mathbb{R})$ and $v, w \in \mathbb{R}^n$

$$(Av) \cdot w = (Av)^\intercal w = v^\intercal A^\intercal w = v^\intercal Aw,$$

and

$$(Aw) \cdot v = v \cdot (Aw) = v^\intercal Aw.$$

Question 5.
Give an example of a formula of an indefinite quadratic form $q\colon \mathbb{R}^2 \to \mathbb{R}$ such that $q((1, 0)) = 9,\ q((0, 1)) = 10$ or prove that it does not exist.

Solution 5.
Let $q((x_1, x_2)) = ax_1^2 + bx_1x_2 + cx_2^2$. Since $q((1, 0)) = a$ and $q((0, 1)) = b$, we have

$$q((x_1, x_2)) = 9x_1^2 + bx_1x_2 + 10x_2^2.$$

Indefinite quadratic form attains positive and negative values. We have, for example

$$q((1, 1)) = 19 + b,$$

hence it is enough to take, for example, $b = -20$. Finally

$$q((x_1, x_2)) = 9x_1^2 - 20x_1x_2 + 10x_2^2.$$